The weakness of the Erdos-Moser theorem under arithmetic reduction

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Reverse mathematics

Almost every theorem is empirically equivalent to one of these five subsystems.

$$\Pi^1_1 - \mathsf{CA}_0 \longrightarrow \mathsf{ATR}_0 \longrightarrow \mathsf{ACA}_0 \longrightarrow \mathsf{WKL}_0 \longrightarrow \mathsf{RCA}_0.$$

We work over the weakest one, RCA_0 which is the fragment of second-order arithmetic whose axioms are the axioms of Robinson arithmetic, induction for Σ^0_1, formulas, and comprehension for Δ^0_1 formulas.

Ramsey theory : RT^n_k

Statement (Ramsey theorem RT^n_k)

For all coloring $f : [\mathbb{N}]^n \to k$, there exists an infinite set $H \subseteq \mathbb{N}$ such that f is constant over $[H]^n$.

First counter example of the aforementioned phenomen : $\mathrm{RCA}_0 < \mathrm{RT}^2_2 < \mathrm{ACA}_0$. The hierarchy collapses after $n = 3$ and is equal to ACA_0. Other versions of RT^n_k are studied.

The Erdős-Moser theorem

Definition

A *tournament* on a domain $D \subseteq \mathbb{N}$ is an irreflexive binary relation $R \subseteq D^2$ such that for every $a, b \in D$ with $a \neq b$, exactly one of $R(a, b)$ and $R(b, a)$ holds.

Alternative definition :

Definition

A *tournament* on a domain $D \subseteq \mathbb{N}$ is an orientation of the complete graph whose set of nodes is D.

Definition

A tournament is *transitive* if for all x, y, z, $R(x, y) \wedge R(y, z) \implies R(x, z)$.

Statement (Erdős-Moser theorem)

EM *is the statement "Every infinite tournament admits an infinite transitive subtournament."*.

EM and RT^2_2

- EM instances, tournaments, can be viewed as 2-colorings of pairs : $f(x,y) = (x < y \wedge R(x,y)) \vee (y < x \wedge R(y,x))$. As such, any f-homogeneous set is in particular a transitive subtournament.

- Jockusch proved that every computable instance of RT^2_2 admits a Π^0_2 solution, while there exists a computable instance of RT^2_2 with no Σ^0_2 solution. These bounds are the same for the Erdős-Moser theorem.

EM and RT^2_2

- Chong proved that the first-order part of Ramsey's theorem for pairs and the Erdős-Moser theorem coincide.
- Most of the known statements implied by RT^2_2 are known to follow from EM over RCA_0.
- Whether EM implies RT^2_2 was open for a long time, before Lerman, Solomon and Towsner answered it negatively.

EM and RT^2_2

Non computable instances make the behaviours vastly differ.

For every function $g : \mathbb{N} \to \mathbb{N}$, there exists an instance of RT^2_2 such that every solution to that instance computes a function dominating g.

Thus, by a theorem of Slaman and Groszek there exists a (non-computable) instance of RT^2_2 such that every solution computes every hyperarithmetic (or equivalently Δ^1_1) set.

EM and RT^2_2

Patey and Wang independently proved that for every non-computable set B and every instance of EM, there exists a solution which does not compute B.

This property of EM is shared with the infinite pigeonhole principle (RT^1_2).

EM is closer to RT^1_2

Monin and Patey proved the following three propositions :

- If B is not arithmetic (resp. hyperarithmetic), then for every set A, there is an infinite subset H of A or $\overline{A}$ such that B is not A-arithmetic (resp. A-hyperarithmetic).
- If B is not Σ^0_n (resp. Δ^0_n), then for every set A, there is an infinite subset H of A or $\overline{A}$ such that B is not $\Sigma^0_n(A)$ (resp. $\Delta^0_n(A)$).
- For every Δ^0_n set A, there is an infinite subset H of A or $\overline{A}$ of low$_n$ degree.

The weakness of the Erdős-Moser theorem under arithmetic reductions

Theorem

If B is not arithmetic, then for every tournament T, there is an infinite transitive subtournament H such that B is not H-arithmetic.

Theorem

Fix $n \geq 1$. If B is not Σ^0_n, then for every tournament T, there is an infinite transitive subtournament H such that B is not $\Sigma^0_n(H)$.

Theorem

Fix $n \geq 1$. Every Δ^0_n tournament T has an infinite transitive subtournament of low$_{n+1}$ degree.

Forcing

A forcing notion

- A forcing notion, a condition : finite strings for Cohen, adding a reservoir for Mathias forcing, ...
- An order over those conditions : extension of finite strings, inclusion of sets, ...

Consider an infinite filter $\mathcal{F}$ and a sufficiently generic set $G_{\mathcal{F}} = \bigcap_{c \in F} [c]$.
For Cohen forcing, this is an infinite decreasing sequence of finite chains, and their union is the sufficiently generic set.

The forcing relation

- Wrong semantic relation : $c \Vdash \varphi(G)$ if $\varphi(G_{\mathcal{F}})$ holds for every filter containing c.
- Correct semantic relation $c \Vdash \varphi(G)$ if $\varphi(G_{\mathcal{F}})$ holds for every **sufficiently generic** filter containing c.
- Syntaxic relation : $(\sigma, X) \Vdash \exists x \psi_e^G(x)$ if $\exists x \psi_e^\sigma(x)$.

The forcing question

The forcing question $c \mathrel{?\!\vdash} \varphi(G)$ asks "does there exist a condition $d \leq c$ such that $d \Vdash \varphi(G)$.

Abstracts from $G_{\mathcal{F}}$ and only talks about conditions : can be simpler computational-wise. For example, for Cohen forcing, whose conditions are only chains, and whose order is computable, the forcing question for a Σ^0_n formula is Σ^0_n : it is *preserving*.

The forcing question needs to be complete : if $c \mathrel{?\!\nvdash} \varphi(G)$, then there exists $d \leq c$ such that $d \Vdash \neg\varphi(G)$".

Σ^0_n-preserving questions

There is no canonical forcing question for a forcing relation, and one needs to construct one fit to force whatever property he wants. Here is an example of such property :

Proposition

Let $(\mathbb{P}, \leq)$ be a notion of forcing with a uniformly Σ^0_n-preserving forcing question. Then for every non-Σ^0_n set B and every sufficiently generic set G for this notion of forcing, B is not $\Sigma^0_n(G)$.

Σ^0_n-preserving questions

Démonstration.

Given a condition $c \in \mathbb{P}$, let $W = \{a \in \mathbb{N} : c \mathrel{?\vdash} \varphi(G, a)\}$. The forcing question is uniformly Σ^0_n-preserving, hence $B \neq W$. Let $a \in B \setminus W \cup W \setminus B$.

- If $a \in W \setminus B$, then by definition, $c \mathrel{?\vdash} \varphi(G, a)$, so by property of the forcing question, there is an extension $d \leq c$ such that $d \Vdash \varphi(G, a)$.
- If $a \in B \setminus W$, then by definition, $c \mathrel{?\nvdash} \varphi(G, a)$. By property of the forcing question, $c \mathrel{?\vdash} \neg\varphi(G, a)$, and by property, there is an extension $d \leq c$ such that $d \Vdash \neg\varphi(G, a)$.

If $\mathcal{F}$ is a sufficiently generic filter, it will contain a condition forcing $\varphi(G, x)$ for an $x \notin B$ or forcing $\neg\varphi(G, x)$ for an $x \in B$ for every Σ^0_n formula $\varphi(G, x)$, hence, letting G be the set induced by $\mathcal{F}$, B will not be $\Sigma^0_n(G)$. □

Combinatorics of EM

Definition

Fix a tournament T over a domain A.

(1) The *interval* (a,b) between $a,b \in A \cup \{-\infty,+\infty\}$ is the set of points $x \in A$ such that $T(a,x)$ and $T(x,b)$ hold.

(2) Given a finite T-transitive subset $F \subseteq A$ and $a,b \in F \cup \{-\infty,+\infty\}$, the interval (a,b) is *minimal* in F if $(a,b) \cap F = \emptyset$.

EM forcing

Any finite T-transitive set F is not necessarily extendible into an infinite solution : suppose there exist some $a, b \in F$ such that $T(a, b)$ holds, but $T(b, x)$ and $T(x, a)$ both hold for cofinitely many x. We shall therefore work with Mathias conditions with some extra structure which will guarantee that σ is extendible into an infinite solution.

Definition

An EM-*condition* for T is a Mathias condition (σ, X) such that

1. for all $y \in X$, $\sigma \cup \{y\}$ is T-transitive ;
2. X is included in a minimal T-interval of σ.

EM forcing

Write $F \to_T E$ if for every $a \in F$ and $b \in E$, $T(a,b)$ holds.

Lemma

Fix an EM*-condition* $c = (\sigma, X)$ *for a tournament* T*, an infinite subset* $Y \subseteq X$ *and a finite* T*-transitive set* $\rho \subseteq X$ *such that* $\max \rho < \min Y$ *and* $[\rho \to_T Y \vee Y \to_T \rho]$*. Then* $(\sigma \cup \rho, Y)$ *is a valid extension of* c*.*

Naive forcing question for Σ^1_0 formulas

Definition

Let $c = (\sigma, X)$ be an EM-condition, n be an integer, and e be a Turing index. Let $c \mathbin{?\vdash} \Phi_e^G(n)\downarrow$ hold if there exists a finite f-homogeneous T-transitive set $\tau \subseteq X$ such that $\Phi_e^{\sigma\cup\tau}(n)\downarrow$.

The tournament T and its limit f have arbitrary complexities : this definition will not yield a preserving question.

Better forcing question for Σ^0_1 formulas

Definition

Let $c = (\sigma, X)$ be an EM-condition, n be an integer, and e be a Turing index. Let $c \mathrel{?\vdash} \Phi_e^G(n)\downarrow$ hold if for every tournament R and every function $g : \mathbb{N} \to 2$, there is a finite g-homogeneous R-transitive set $\tau \subseteq X$ such that $\Phi_e^{\sigma \cup \tau}(n)\downarrow$.

The over-approximation of the tournament and its limit actually reduce the complexity thanks to a compactness argument :

$c \mathrel{?\vdash} \Phi_e^G(n)\downarrow$ if there exists some threshold t such that for every tournament R over $\{0, \dots, t\}$ and every function $g : \{0, \dots, t\} \to 2$, there is a finite g-homogeneous R-transitive set $\tau \subseteq X$ such that $\Phi_e^{\sigma \cup \tau}(n)\downarrow$.

Forcing Π^2_0 formulas

Not as easy : conditions do not strongly force now, but it is dense to force a collection of Σ^0_1 formulas.

Stating this density has varying complexity depending on the notion of forcing. It is simple enough for Cohen forcing, but not for Mathias forcing. The following lemma proves this approach fails for Mathias forcing :

Lemma (Folklore)

The set $\emptyset''$ is $\Pi^0_2(G_{\mathcal{F}})$ for every sufficiently generic filter $\mathcal{F}$ for Mathias forcing with computable reservoirs.

The idea is that reservoirs are too sparse.

Partition regular and large classes

Definition

A class $\mathcal{L} \subseteq 2^\omega$ is *partition regular* if :

- $\mathcal{L}$ is non-empty,
- for all $X \in \mathcal{L}$, if $X \subseteq Y$, then $Y \in \mathcal{L}$,
- for every integer k, for every $X \in \mathcal{L}$, for every k-cover $Y_1, Y_2, \dots Y_k$ of X, there exists $i \leq k$ such that $Y_i \in \mathcal{L}$.

Definition

A class $\mathcal{L} \subseteq 2^\omega$ is *large* if :

- for all $X \in \mathcal{L}$, if $X \subseteq Y$, then $Y \in \mathcal{L}$,
- for every integer k, for every k-cover $Y_1, Y_2, \dots Y_k$ of ω, there exists $i \leq k$ such that $Y \in \mathcal{L}$.

Partition regular and large classes

Lemma

Let $(\mathcal{P}_n)_{n\in\omega}$ *be a decreasing sequence of large classes. Their intersection* $\bigcap_{n\in\omega}\mathcal{P}_n$ *is again large.*

Lemma

Let $\mathcal{A}$ *be a* Σ^0_1 *class. The sentence "*$\mathcal{A}$ *is large" is* Π^0_2*.*

Definition

For every large class $\mathcal{P}$, let $\mathcal{L}(\mathcal{P})$ denote the largest partition regular subclass of $\mathcal{P}$.

Lemma

For every set $C \subseteq \omega^2$*, there exists* $D \leq_T C$ *such that* $\mathcal{U}_D^{\mathcal{M}} = \mathcal{L}(\mathcal{U}_C^{\mathcal{M}})$*.*

First theorem

Theorem

If B is not arithmetic, then for every tournament T, there is an infinite transitive subtournament H such that B is not H-arithmetic.

The forcing question to decide $\Sigma_2^0(G)$ formulas is too big. However, since it is still arithmetic, this is not an issue and we make it work, since B is not arithmetic.

Proving second theorem

Theorem

Fix $n \geq 1$. If B is not Σ^0_n, then for every tournament T, there is an infinite transitive subtournament H such that B is not $\Sigma^0_n(H)$.

An added difficulty is that now we have to find a way to reduce the complexity of the forcing question at the top level. We build a new forcing notion and a new forcing question to fix this issue.

Proving third theorem

Theorem

Fix $n \geq 1$*. Every* Δ^0_n *tournament* T *has an infinite transitive subtournament of low*$_{n+1}$ *degree.*

We prove this result by constructing our set effectively. This comes with its fair share of technical difficulties but is quite standard.

Thank you for listening

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