Affine completeness of some free binary algebras

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September 10, 2024

Outline

1. Congruences, congruence preservation, affine completeness
2. Contributions of the present work

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Congruences

A **CONGRUENCE** on algebra $\mathcal{A} = \langle A, \star \rangle$ algebra with operation $\star$. is an equivalence relation $\sim$ on A which is compatible with the operation, i.e.,

$$a \sim a' \text{ and } b \sim b' \implies a \star b \sim a' \star b'$$

Congruence preserving functions

Definition

$f : A^n \to A$ is congruence preserving iff, for any congruence $\sim$ on A, $\forall x_1, \ldots, x_n, y_1, \ldots, y_n \in A$
$\bigwedge_{i=1}^{i=n} x_i \sim y_i \implies f(x_1, \ldots, x_n) \sim f(y_1, \ldots, y_n)$

Example: Polynomial functions are CP (congruence preserving) .

$P = \{$polynomials with variables $x_1, \ldots, x_n\}$ is defined by

- $A \cup \{x_1, \ldots, x_n\} \in P$
- $t, t' \in P \implies (t \star t') \in P$

Not all CP functions are polynomial ,,,,

Affine completeness

CP= congruence preserving.
Non polynomial CP functions on $\langle \mathbb{N}, + \rangle$:

$$g(x) = \frac{\Gamma(1/2)}{2 \times 4^x \times x!} \int_1^\infty e^{-t/2}(t^2-1)^x dt$$

[CGG15] *Newton representation of functions over natural integers having integral difference ratios*, Int. Jour. Numb. Th., (2015).

Definition (affine complete algebra)

An algebra is affine complete iff:
for all f: f CP $\iff$ f polynomial.

Hence: $\langle a^*, \cdot \rangle \approx \langle \mathbb{N}, + \rangle$ is not affine complete.

Affine completeness in a non-commutative algebra

On the algebra of words with concatenation, $\mathcal{S} = \langle \Sigma^*, \cdot \rangle$, for f unary

Theorem (In the Free monoid -CGG)

If $|\Sigma| \geq 3$*:* f *CP* $\iff$ $f : x \mapsto w_0 x w_1 x w_2 \cdots x w_k$, $k \in \mathbb{N}$, $w_0, w_1, \ldots, w_k \in \Sigma^*$.

Similar Theorem for n-ary functions.

[CGG] *CP functions on free monoids*, Alg.Univ. (2017).

Summary: The free monoid $\langle \Sigma^*, \cdot \rangle$ is not affine complete if $|\Sigma| = 1$ and is affine complete if $|\Sigma| \geq 3$.

$$|\Sigma| = 2 \text{ ??}$$

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1 Congruences, congruence preservation, affine completeness

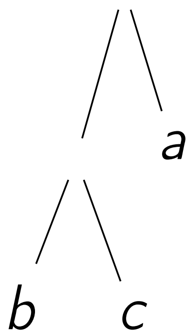
2 Contributions of the present work

Algebras of binary trees with labelled leaves

Σ an alphabet

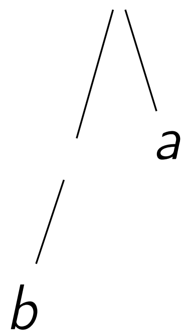
$\mathcal{T}_{\mathcal{C}}$: complete trees generated by Σ, $\star$

example



$\mathcal{T}$: non complete trees generated by $\emptyset$, Σ, $\star$

example



New results of the present work

Theorem

The algebras of trees $\mathcal{T}$ (or complete trees $\mathcal{T}_{\mathcal{C}}$) are affine complete if $|\Sigma| \geq 1$.

Theorem

The algebra of words with concatenation, $\mathcal{S} = \langle \Sigma^*, \cdot \rangle$, $|\Sigma| \geq 2$ *is affine complete.*

Similar proofs: by induction on arity n of f.
Case $n = 0$ obvious. Will sketch proof for $n = 1$.
Going from n to $n + 1$: identical.
We use only some special "magic" congruences. From now on f unary CP on $\Sigma^* = \{a, b\}^*$

Proof idea for f unary on $\Sigma^* = \{a, b\}^*$

f CP $\overset{??}{\Longrightarrow} \exists k \in \mathbb{N}, w_0, w_1, \ldots, w_k \in \Sigma^*\colon\ f(x) = w_0 x w_1 x w_2 \cdots x w_k.$

- length of $u = |u|$ = number of letters in u: e.g., $|axbxa| = 5$
- $u \sim v$ iff $|u| = |v|$ is a congruence.
- **degree of polynomial** P**:** = number of "x" in $P(x)$

$P(x) = axbxa$, degree($P(x)$)=2, $|P(u)| = |aubua| = 2|u| + 3$ hence $|P(u)|$ is an affine function of $|u|$.

- for all CP functions is $|f(u)|$ is an affine function of $|u|$??

Lemma (Degree of a CP function)

If $f : \Sigma^* \longrightarrow \Sigma^*$ is CP then $\exists k$ called the degree of f such that:

$$|f(u)| = k|u| + |f(\varepsilon)|$$

.

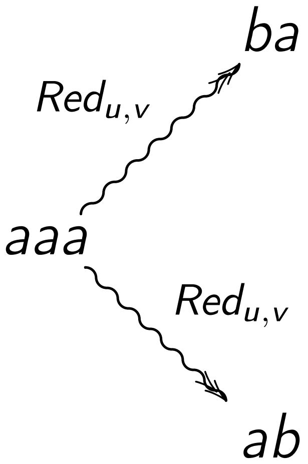
Special congruences on Σ^*

$u, v \in \Sigma^*$ $\sim_{u,v}$ **congruence** generated by $u \sim_{u,v} v$

$|u| > |v|$, $w \in \Sigma^*$ is $\sim_{u,v}$-**reducible** iff u occurs in w.

Define reduct $Red_{u,v}(w)$ by replacing u by v in w as much as possible: $w \overset{Red_{u,v}}{\rightsquigarrow} Red_{u,v}(w)$.

Example: $u = aa$, $v = b$, $w = aaa$, $aa \overset{Red_{u,v}}{\rightsquigarrow} b$



Reduct non-unique: because
u **self-overlaps in** w **to be avoided**

Magic congruences on Σ^*

$$\sim_{\tau,v} \quad \text{with } \tau \in \mathcal{T} = \{a^n bab^n \mid n > 1\}, \text{and } |v| < |\tau|$$

Each congruence class has a unique canonical shortest representative, i.e.,

$$u \sim_{\tau,v} u' \iff u \overset{Red_{\tau,v}}{\rightsquigarrow\!\!\!\twoheadrightarrow} w \overset{Red_{\tau,v}}{\twoheadleftarrow\!\!\!\leftarrow\!\!\!\!\rightsquigarrow} u'$$

Magic congruences: Each u has a unique decomposition with u_i's τ-irreducible $u = u_0 \tau u_1 \tau \ldots \tau \ldots \tau u_m$.

Apply to $u = f(\tau)$: $f(\tau) = u_0 \tau u_1 \tau \ldots \tau \ldots \tau u_m$.

Define a Polynomial

$$P_{f,\tau}(x) = Red_{\tau,x}(f(\tau)) = u_0 x u_1 x \ldots x \ldots x u_m$$

Partial polynomiality of CP functions

Lemma ($LemP_{f,\tau}$)

The polynomial $P_{f,\tau}(x) = Red_{\tau,x}(f(\tau))$ satisfies:
$P_{f,\tau}(\tau) = f(\tau)$, *degree*$(P_{f,\tau}) \leq$ *degree*(f) *and*
$\forall v$: $|f(v)| < |\tau| \Longrightarrow f(v) = Red_{\tau,v}(P_{f,\tau}(v))$.

Proof $|f(v)| < |\tau|$ implies $f(v) = Red_{\tau,v}(f(v))$

$$f(v) \sim_{\tau,v} f(\tau) \quad = \quad P_{f,\tau}(\tau) \sim_{\tau,v} P_{f,\tau}(v)$$

$$Red_{\tau,v} \downarrow \qquad\qquad\qquad \downarrow Red_{\tau,v}$$

$$f(v) \quad = \quad Red_{\tau,v}(P_{f,\tau}(v))$$

Theorem

If degree($P_{f,\tau}$)= degree(f) then $\forall v$ with $|f(v)| < |\tau|$,
$f(v) = P_{f,\tau}(v)$.

Proof Same degree $\Longrightarrow$ same length $\Longrightarrow |P_{f,\tau}(v)| = |f(v)| < |\tau| \Longrightarrow P_{f,\tau}(v) = Red_{\tau,v}(P_{f,\tau}(v))$.

Fundamental lemma

f CP of degree k. Goal : $degree(P_{f,\tau}) = degree(f) = k$

Lemma (LemF)

Let $|\tau| > |f(a)|$. If $degree(P_{f,\tau}) < k$ there exists $\ell \in \{a, b\}$ such that $|Red_{\tau,\ell}(P_{f,\tau}(\ell))| > |\tau|$.

Consequence of Lemma

$$\left.\begin{array}{r} |f(\ell)| = |f(a)| < |\tau| \\ \text{By Lem}P_{f,\tau}\text{: } f(\ell) = Red_{\tau,\ell}(P_{f,\tau}(\ell)) \\ \text{By LemF: } degree(P_{f,\tau}) < k \implies |Red_{\tau,\ell}(P_{f,\tau}(\ell))| > |\tau| \end{array}\right\} \textit{Contradiction}$$

$|\tau| > |f(a)| \implies degree(P_{f,\tau}) = degree(f) \implies$
if $|f(v)| < |\tau|$, $f(v) = P_{f,\tau}(v)$ polynomial

Example for proof of fundamental Lemma

The proof relies on combinatorial properties of words in $\mathcal{T}$.
If $|f(u)| = 2|u|$, $\tau = a^2bab^2$, $|\tau| = 6$, and $degree(P_{f,\tau}) = 1$, then $degree(f) = 2$, $P_{f,\tau}(x) = wxw'$, w, w' τ-irreducible, $ww' = \tau$.
Hence 5 possibilities for $P_{f,\tau}$.

$P_{f,\tau}(x)$	$axabab^2$	a^2xbab^2	a^2bxab^2	a^2baxb^2	a^2babxb
$P_{f,\tau}(a)$	a^3bab^2 $= a\tau$	a^3bab^2 $= a\tau$	$a^2ba^2b^2$ τ-irred.	$a^2ba^2b^2$ τ-irred.	a^2babab τ-irred.
$P_{f,\tau}(b)$	$ababab^2$ τ-irred.	$a^2b^2ab^2$ τ-irred.	$a^2b^2ab^2$ τ-irred.	a^2bab^3 $= \tau b$	a^2bab^3 $= \tau b$

For each of these five possibilities, at least one of the two words $P_{f,\tau}(a)$, $P_{f,\tau}(b)$ is of length $7 > 6 = |\tau|$ and is $|\tau|$-irreducible.

Polynomiality of CP functions

Consequence of the fundamental lemma:

Theorem

For any v, $\exists n$ such that : $|\tau| > n \Longrightarrow f(v) = P_{f,\tau}(v)$.

Moreover

Lemma

P, Q polynomials such that: $P(a) = Q(a)$ *and* $P(b) = Q(b)$ *then* $P = Q$.

Consequence: All $P_{f,\tau}$ are equal for τ large enough and their common value P satisfies $f = P$. Hence any unary CP function is polynomial. The n-ary case is done by a simple induction, and $\{a, b\}^*$ is affine complete.

CONCLUSION

Simpler proof for tree algebras: because of the unique decomposition of trees, there are no overlapping problems and any tree can play the role of τ.

Problems

– What about labelled non binary trees
– Characterize affine complete binary free algebras